Home Work III

1. Let a, b be positive numbers. Show that a < b iff a^2 < b^2. (Is it true if a < 0?)
2. Let b, c be real numbers. Show that the quadratic equation x^2 -2bx +c = 0 is solvable in R (in the sense that it has one or more real solution) iff b^2 -c is nonnegative. (Hint: make use "perfect square")
3. Let z, a be positive be such that z^2 > a. Show that there exists a natural number m such that (1/m) < z and (z - 1/m)^2 > a.
4. Let z, a be positive be such that z^2 < a. Show that there exists a natural number` n such that (z + 1/n)^2 < a.

5* Let a be positive and let B consist of all positive real numbers x with x^2 > a. Show that B is nonempty and z: = inf B exists in R. Show further that z^2 = a.

6. Let a > 0. Similar to Q5 but use suitable set A and sup A to show the existence of the positive square root of a.

7. Show

$$\big|\,|x|-|y|\,\big| \leq |x-y| \leq |x|+|y| \quad \forall\, x,y \in \mathbb{R}$$

and describe when the inequality is strict. Sketch/interpret your results geometrically.

8*. Solve the inequality system (in the sense to identify the "solution set" A, the set consisting of all x satisfying the inequalities):

$$4 < |x+2| + |x-1| \leq 5.$$

(A = the union of (-3, -5/2) and (3/2, 2]). Hint: subdivide into subintervals in order to remove the absolute value sighs

9*. Let y and t be real numbers. Show that following assertions:

(a) If $|t| < 10$ and $|y-t| < 3$ then $|y| < 13$;

(b) If $t \neq 0$ and $|y-t| < \frac{|t|}{2}$ then $\frac{|t|}{2} < |y|$.

10. Show by MI the binomial formula

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{k} a^{n-k} b^k + \cdots + \binom{n}{n-1} a b^{n-1} + b^n$$

Hint: Use $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

and, for a > 0 show that

$$(1+a)^n \geq 1 + na$$

and

$$(1+a)^n \geq \frac{n(n-1)(n-2)}{3!} a^2 \quad \forall\, n \geq 3$$

so $\frac{n^2}{(1+a)^n} \to 0$ as $n \to \infty$. $\left[\text{Similarly } \frac{n^{1000}}{(1.0001)^n} \to 0\right]$